Appendix: Formal analysis of the effects of demand variability of costs in a model with long-term and short-term contracts.

to accompany:

Baker, Phibbs, Guarino, Supina, and Reynolds, "Within-Year Variation in Hospital Utilization and its Implications for Hospital Costs," Journal of Health Economics

Maintaining notation from the main body of text, suppose that on any given day the hospital faces demand z, where z is a random variable with distribution function $G(z)$ and corresponding probability density function $g(z)$, with G and g known to the hospital. The turnaway probability is a, which defines the maximum level of demand it has to be prepared to face: $\bar{z} = G^{-1}(1-a)$, and hospitals need then provide buildings, beds, and other fixed or quasi-fixed factors sufficient to meet that level of demand.

Suppose that in addition to fixed inputs, hospitals require a single variable input to care for patients (e.g. labor), which they can purchase in two markets. Before any specific values for demand are realized, hospitals can write long-term contracts that commit the hospital to pay a certain price per unit and buy a certain amount of the input each day whether the full amount is needed or not. Alternatively, hospitals can wait until demand for the day is realized and purchase needed inputs on the spot market. Denote by x_l and x_s the number of inputs the hospital buys with long-term and short-term spot contracts, respectively, and by w_l and w_s the price per unit that the hospital pays for each, with $w_s > w_l$.

Let f denote the (invertible) production function, so that $x=x_l+x_s$ units of variable input can meet z units of demand: $z=f(x; k)$, where $f' >0$, $f'' <0$, and k denotes previously determined level of fixed factors needed to satisfy the turnaway probability constraint. The hospital seeks to minimize expected variable input costs, with the restriction that for realizations of demand higher than $\bar{z}$, the hospital need only provide the amount of input needed to produce $\bar{z}$. The expected cost function that the hospital seeks to minimize is then given by

$$\int_{-\infty}^{f(x_l)} w_l x_l g(z)dz + \int_{f(x_l)}^{\bar{z}} [w_l x_l + w_s x_s(z)]g(z)dz + \int_{\bar{z}}^{\infty} [w_l x_l + w_s x_s(\bar{z})]g(z)dz \qquad \text{(A-1)}$$

where x_s is written as a function of z to reflect the fact that the number of units of input purchased in the spot market will vary with the realizations of demand. We implicitly assume throughout that x_l will always be set to fall within rational bounds, such that $0 \leq f(x_l) \leq \bar{z}$.

The first term is spending on inputs acquired through long-term contracts, which the hospital must purchase for all levels of observed demand below $f(x_l)$. (For completeness and convenience, this integral includes the interval $(-\infty, 0)$. In this range, observed demand would be 0, but the hospital would still have to purchase the inputs for which it had contracted.) The second term captures the interval on which the hospital purchases x_l inputs plus some inputs in the spot market to meet realized demand. The last term captures levels of demand above the hospital's planned turnaway point, at which the hospital has only to provide inputs to meet demand at the turnaway point since we assume that they have made other fixed capacity decisions beforehand that preclude taking care of additional patients.

Simplifying and using the fact that $x_s(z) = f^{-1}(z) - x_l$ for any level of z above $f(x_l)$, and $x_s(z)$=0 otherwise, we can write this as

$$w_l x_l + \int_{f(x_l)}^{\bar{z}} w_s [f^{-1}(z) - x_l] g(z) dz \ + \ a \ w_s [f^{-1}(\bar{z}) - x_l] \qquad \text{(A-2)}$$

The cost minimizing level of inputs purchased in the long-term market is obtained by maximizing (A-2) with respect to x_l. The first order condition is:

$$w_l - \int_{f(x_l)}^{\bar{z}} w_s g(z) dz \ + \ a \ w_s \ = \ 0. \qquad \text{(A-3)}$$

From which the optimal value for x_l can be obtained:

$$x_l^* = f^{-1}(G^{-1}(\frac{w_s - w_l}{w_s})) \qquad \text{(A-4)}$$

Since G^{-1} and f^{-1} are increasing, this produces the intuitive result that as the price of spot-market inputs rises relative to the price of long-term contracts, hospitals will increase the number of inputs they buy in the long-term contract market and decrease the number they buy in the spot market.

The relationship between the variance of G and spending for inputs is our main concern. Note that for G normal with mean m and variance s^2,

$$G^{-1}(k) = \Phi^{-1}(k) s + m \qquad \text{(A-5)}$$

for any critical value k, where Φ is the standard normal cumulative distribution function.

Substituting this into (A-4), we note that $\partial x_l^* / \partial \sigma < 0$ when $(w_s-w_l)/w_s < 0.5$ and

$\partial x_l^* / \partial \sigma > 0$ when $(w_s-w_l)/w_s > 0.5$. That is, the optimal allocation of inputs might increase

or decrease with changes in variance, depending on the relative input prices.

The net effect of an increase in variance is always to increase costs. Substituting x_l*

into (A-2) gives the optimal level of total spending on inputs:

$$w_l x_l^* + \int_{f(x_l^*)}^{\bar{z}} w_s (f^{-1}(z) - x_l^*) g(z) dz \ + \ \alpha \, w_s [f^{-1}(\bar{z}) - x_l^*] \qquad \text{(A-6)}$$

or, collecting the x_l* terms,

$$x_l^* \left[w_l - [1 - G(f(x_l^*))] w_s \right] + \int_{f(x_l^*)}^{\bar{z}} w_s f^{-1}(z) g(z) dz \ + \ \alpha \, w_s f^{-1}(\bar{z}) \qquad \text{(A-7)}$$

Since $x_l^* = f^{-1}(G^{-1}((w_s-w_l)/w_s)))$, the first term is

$$x_l^* (w_l - w_s (1 - \frac{w_s - w_l}{w_s})) \ = \ 0 \qquad \text{(A-8)}$$

That is, spending on long-term contract assets and expected spending on spot market purchases

are equalized at the optimal value of x_l*. We are left with the simpler equation to manage

$$\int_{f(x_l^*)}^{\bar{z}} w_s f^{-1}(z) g(z) dz \ + \ \alpha \, w_s f^{-1}(\bar{z}) \qquad \text{(A-9)}$$

Which we can write in terms of *G* as

$$w_s\left[\int_{G^{-1}(w)}^{G^{-1}(1-a)} f^{-1}(z)g(z)dz \quad + \quad a\ f^{-1}(G^{-1}(1-a))\right] \tag{A-10}$$

where w is defined as $(w_s\text{-}w_l)/w_s$ to simplify notation. G and g are the cdf and pdf of a normal distribution with mean m and variance s^2. A change of variables in the integral replaces *g* with the standard normal and simplifies later calculations. Let $y = (z-m)/s$ so $z = s\ y + m$. Then $g(z) = f(y)$ where f is the standard normal pdf. Changing variables, we have

$$w_s\left[\int_{\frac{1}{s}\left[G^{-1}(w)-m\right]}^{\frac{1}{s}\left[G^{-1}(1-a)-m\right]} f^{-1}(sy+m)\,f(y)\,dy \ + \ a\ f^{-1}(G^{-1}(1-a))\right] \tag{A-11}$$

Writing in terms of Φ, the standard normal cdf, we have:

$$w_s\left[\int_{\Phi^{-1}(w)}^{\Phi^{-1}(1-a)} f^{-1}(sy+m)\,f(y)\,dy \ + \ a\ f^{-1}[s\ \Phi^{-1}(1-a)+m]\right] \tag{A-12}$$

The derivative of (A-12) with respect to s is

$$w_s\left[\int_{\Phi^{-1}(w)}^{\Phi^{-1}(1-a)} f^{-1\,\prime}\ y f(y)dy \ + \ a\ f^{-1\,\prime}\ \Phi^{-1}(1-a)\right] \tag{A-13}$$

which is positive since w_s is positive, and both terms in brackets can be signed to be positive. This quantity increases as the price of spot-market inputs diverges from the price of long-term contract inputs, and is invariant to changes in the mean of *G*.

This establishes the intuitive conclusion that increasing variability in demand will make it more difficult for hospitals to efficiently contract for labor or other inputs and will tend to drive up costs, in addition to the well-known conclusion that increases in variance contribute to increases in costs by requiring additional standby capacity.